LETTER TO THE EDITOR

The fractal dimension of cluster perimeters generated by a kinetic walk

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Abstract. The fractal dimension d_G of cluster perimeters generated by a recently proposed 'butterfly' growth walk is considered. In the long-range limit of the walk on a percolation cluster, d_G appears to be equal to the fractal dimension of the singly connected bonds: $d_G = 1/\nu$. The new relation for chemical dimension d_l is proposed: $d_l = d_f/(d_f - d_G)$. In the short-range limit the 'butterfly' walk on a Euclidean lattice appears to be in the same universality class as a random walk. The dynamic aspect of the growth walk is discussed and the continuously tunable spectral dimension is obtained. Both short- and long-range limits of this diffusion process are different from a random diffusion on percolation.

The structure of clusters generated by kinetic walks is currently a subject of intensive study (see, for example, Kremer and Lyklema 1984, Majid *et al* 1984, Peliti 1984, Ziff *et al* 1984). The main point of interest is the fractal dimensionality d_f of the trace left by a walk with a one-step probability which depends on the past history. A new type of kinetic walk was introduced recently (Bunde *et al* 1984) which visits only the perimeter sites of the cluster generated by the walk. These perimeter sites through which the cluster continues growing were termed growth sites (Leyvraz and Stanley 1983). At each step of the walk the choice of the next growth site to be visited is performed according to the probability $P(r) \sim 1/r^\alpha$ that depends on the distance r from the current visited site. The parameter α (positive or negative) governs the effective attraction or repulsion between the growth sites. Thus, the 'butterfly' walk makes steps of varying length, limited by the size of a growing cluster. The main property of interest for this type of walk is not the fractal dimensionality d_f of the grown cluster which (except for the case of $\alpha \to -\infty$ on a Euclidean lattice) coincides with the fractal dimensionality of the substrate. Rather, the fractal dimensionality d_G of the growth sites forming the growth perimeter of the cluster is of interest. The number of growth sites G scales with the average cluster radius R as

$$G \sim R^{d_G}. \tag{1}$$

The 'butterfly' walk was studied by Bunde *et al* (1984) on the percolation cluster in $d = 2$ at the threshold p_c. An interesting crossover behaviour was found for d_G, namely

$$d_G = \begin{cases} 0.76 \pm 0.03 & \alpha \leq 2 \\ 1.04 \pm 0.04 & \alpha \geq 8. \end{cases}$$

These two limits correspond, respectively, to the long-range and the short-range

'butterfly' walk. The long-range walk includes the case of $\alpha = 0$ which corresponds to the Eden model on percolation clusters (see also Family and Vicsek 1985).

In this letter the crossover phenomena occurring in the 'butterfly' model are studied further. A conjecture is proposed identifying d_G/d_f for the long-range limit with the known percolation exponent σ. The 'butterfly' walk is performed on a Euclidean lattice ($p = 1$) and it is found that in the short-range limit $G \sim s/\ln s$, corresponding to the growth perimeter of a random walk. I also introduce the dynamic aspect to the 'butterfly' walk and find the crossover in the spectral dimension as well.

Let us look at the way the large cluster perimeter P scales with the number of cluster sites s at the percolation threshold. The cluster perimeter P of the finished cluster is the number of blocked neighbours to cluster sites. Note that the growth perimeter G is defined for the cluster which is still growing and, therefore, is equal to the number of all nearest neighbours to cluster sites minus blocked sites. It is known (Stauffer 1979) that

$$P = s \frac{1 - p_c}{p_c} + As^\sigma \qquad (2)$$

where $\sigma = 1/\nu d_f$ and ν is a correlation length exponent; A is a constant. The statistical weights of both the finished clusters and the clusters which are still growing are proportional to

$$\sum_s sn_s \sim s^{2-\tau}$$

where n_s is a number of s-site clusters per site and τ is a percolation exponent; $\tau \simeq 2.055$ in $d = 2$). It is natural to assume, therefore, that (2) represents the scaling form for perimeters of very large growing clusters as well. One can also perceive that such correspondence should hold only for the long-range limit of the 'butterfly' walk. This can be seen most clearly in the case of $\alpha = 0$ when the growth sites are chosen at random, closely resembling the Leath (1976) method of cluster growth which results in (2). Another comparison can be made with the growth variation of the Alexandrowicz (1980) method leading to the same long-range value of d_G as a 'butterfly' (Family and Vicsek 1985). On the other hand, the short-range limit leads to a different growth pattern where the growth sites chosen tend to be close to each other and, consequently, the growth perimeter scales in a different way. The first term in (2) can be easily identified with the number of blocked sites in the growing cluster of size s. Therefore, for a large enough growing cluster the number of growth sites in a long-range limit should be proportional to the 'excess' perimeter

$$G \sim s^\sigma \sim R^{1/\nu}. \qquad (3)$$

Comparing (1) and (3) one gets

$$d_G = 1/\nu. \qquad (4)$$

Numerically, this is in a very good agreement with the long-range limit value for d_G found by Bunde *et al* (1984) since $\nu = \frac{4}{3}$ and $d_G = \frac{3}{4}$ for $d = 2$.

The simple argument presented here gives the same value for d_G in the long-range limit as does the argument given by Bunde *et al* (1984), but it is more general. It should apply in all dimensions. It certainly holds for the Cayley tree where $d_G/d_f = \frac{1}{2}$ (Leyvraz and Stanley 1983) and $\sigma = \frac{1}{2}$. There appears to be no difference between the long- and short-range limit 'butterfly' and the 'ant' for the Cayley tree. Preliminary

results for $d = 3$ (simple cubic lattice) also seem to agree with the prediction $d_G/d_f = \sigma \simeq 0.45$.

There is another meaning to the result $d_G = 1/\nu$. The fractal dimension d_G of growth sites G in the long-range limit appears to be the same as the fractal dimension of singly connected bonds L (Coniglio 1982). This might lead to the following scaling behaviour of the suitably averaged growth perimeter below the percolation threshold

$$G \sim L \sim |p - p_c|^{-1} \tag{5}$$

which is currently being checked numerically.

Another way to interpret the long-range limit was proposed by Leyvraz (1984). Suppose that the loops can be neglected. Then the growth perimeter G is a fractal cut in the 'chemical distance space' (Havlin 1984)

$$G \sim l^{d_l - 1} \sim s^{(d_l - 1)/d_l}. \tag{6}$$

Here l stands for the 'chemical length' (the minimum cluster path between two points) and d_l is the 'chemical' fractal dimension defined by $s \sim l^{d_l}$. Now, (6) combined with (3) leads to the following scaling relation

$$d_l = \frac{1}{1 - \sigma} = \frac{d_f}{d_f - d_G} \tag{7}$$

connecting the chemical dimension d_l to the fractal dimension of the percolation cluster d_f and fractal dimension of growth sites $d_G = 1/\nu$. This relation was indirectly predicted by Havlin and Nossal (1984) based on their numerical observation. Note that since $d_l = d_f/d_{min}$ where d_{min} is the fractal dimension of the minimum path, one also gets

$$d_{min} = d_f - d_G. \tag{8}$$

Numerically, this leads in $d = 2$ to $d_l = 91/55 \simeq 1.65$ and $d_{min} = 55/48 \simeq 1.15$ (but see Grassberger 1985 for a considerably larger numerical value of d_l). For $d = 3$ one gets $d_l \simeq 1.82$ and $d_{min} \simeq 1.37$. We see that the fractal dimension of the growth perimeter d_G is an important quantity and might reduce the number of 'independent' fractal dimensions.

The 'butterfly' walk on the Euclidean lattice ($p = 1$) is of major interest now since it gives an opportunity to study the crossover in d_G without additional complications: the number of growth sites G is equal to cluster perimeter for $p = 1$. The case of $\alpha = 0$ reduces to the Eden model with $d_G = 1$ (Peters *et al* 1979). Another simple case is $\alpha = -\infty$: the 'butterfly' chooses the site which is at maximal distance from the current site. The resulting cluster is a straight line: $d_f = 1$ and $d_G = 1$. For intermediate values of α, I observe elongated clusters (see figure 1) which seem to represent transient behaviour from $d_G/d_f = \frac{1}{2}$ (Eden model) to $d_G/d_f = 1$ ($\alpha = -\infty$). The effective value of d_G/d_f for $\alpha = 16$, for example, is about 0.6.

Much more interesting is the short-range limit at $p = 1$. Let us look in more detail at the short-range 'butterfly' behaviour. The value of G does not change for $\alpha > 16$. We are dealing here with a well defined space-filling walk ($d_f = 2$). Namely, the walk goes to the nearest neighbouring sites, except when it falls into traps (all nearest neighbours are occupied). In this case, the 'butterfly' chooses the nearest perimeter site. Thus, the short-range 'butterfly' does not avoid traps as does the infinitely growing SAW (Kremer and Lyklema 1984) and tries to avoid flying over visited sites more efficiently than the 'true' SAW (Amit *et al* 1983). As a result it forms compact clusters
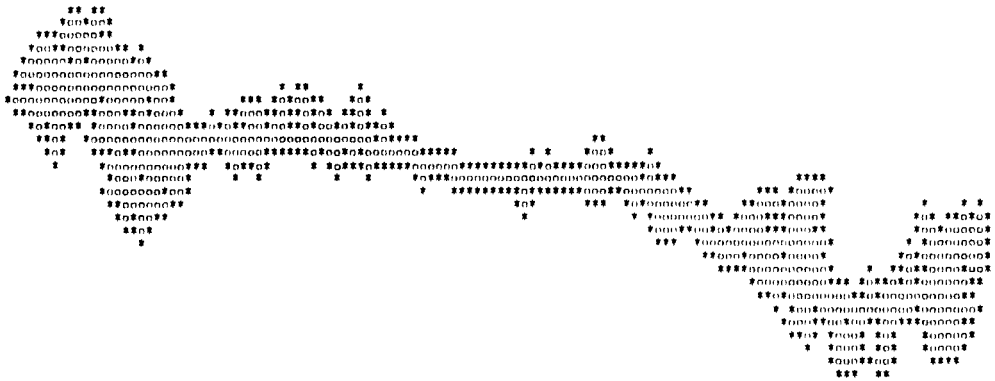
Figure 1. The cluster of 500 sites formed by a $\alpha = -32$ 'butterfly' on a Euclidean lattice, $\bigcirc$, a visited site; *, a growth site.

with the growth perimeter: $G \sim s^{0.85}$ (see figure 2). However, the apparent exponent 0.85 can be interpreted in a different way. Namely, a fit $G \sim s/\ln s$ is of roughly the same numerical quality. Note that for a random walk in $d = 2$ one finds that $s \sim N/\ln N$ and, accordingly, $G \sim s/\ln s$ (Rammal and Toulouse 1983). Thus, I find that the growth perimeter of the short-range 'butterfly' walk behaves as the one for a random walk. This, however, is not the case for a walk on a percolation cluster where the random walk gives $d_G \simeq 0.93$ (Stanley *et al* 1984) and the short-range 'butterfly' walk, $d_G \simeq 1$. This discrepancy is very intriguing and yet unexplained. One is tempted to summarise the numerical and scaling results by suggesting that for $d = 2$

$$d_G = \begin{cases} 1 & \alpha < 2 \\ 2 & ?\alpha > 16 \end{cases} \qquad p = 1$$

$$d_G = \begin{cases} \frac{3}{4} & \alpha < 2 \\ 1.04 \pm 0.04 & ?\alpha > 16 \end{cases} \qquad p = p_c. \tag{9}$$

I conclude with a discussion of the dynamic aspect of the 'butterfly' walk. The dimension d_w of this walk (where $(\text{time}) \propto (\text{distance})^{d_w}$) is identical to d_f by construction: there is no revisiting of the sites. This implies a rescaling of the time for the real diffusion process since we only increase time t by 1 when the cluster size s grows by 1, purging the time of revisiting. Thus, the 'butterfly' moves in an artificial time $t = s$. We can, however, rescale the time back to normal

$$s \sim t^{d_f/d_w} \qquad G \sim t^{d_G/d_w}. \tag{10}$$

It is as if the random walk proceeds normally until it meets a growth site. At this moment decision to occupy it or not is made according to a 'butterfly' probability $P(r) \sim 1/r^\alpha$. If the growth site is not occupied, the revisiting continues until the next growth site. I suggest that this type of walk is called a growth-limited diffusion process. To model the diffusion successfully one should also preserve the time correlations between the changes ΔG in the number of growth sites inherent in the diffusion process. This is very hard to do since not much is known about these correlations (Aharony *et al* 1985). I suggest a simple way of introducing the dynamic aspect into the 'butterfly' walk. Since the choice of the next visited growth site is already made in the framework

Figure 2. The cluster of 1500 sites formed by a $\alpha = +32$ 'butterfly' on a Euclidean lattice. Only the growth sites, *, are shown.

of the model, let us assume that the probability of access to the growth site is proportional to the growth perimeter G (Rammal and Toulose 1983)

$$\frac{ds}{dt} \sim \frac{G}{s}. \tag{11}$$

This equation seems to work for the 'ant in the labyrinth' (Leyvraz and Stanley 1983). It should also be fine for the $\alpha = 0$ 'butterfly' which choses growth sites at random. If one assumes it to hold for all α one easily gets combining (11), (10) and (2) (Stanley 1984)

$$\frac{d_w}{d_f} = 2 - \frac{d_G}{d_f}. \tag{12}$$

This immediately leads, using (9) and $d_f = 91/48$, to an interesting crossover for $d = 2$ in a spectral dimension $d_s = 2d_f/d_w = 1 + d_G/d_w$ from $d_s = 91/73 \simeq 1.25$ (long-range limit) to $d_s = 1.38 \pm 0.02$ (short-range limit) with the value for the 'ant' $d_s \simeq 1.32$ (Stanley *et al* 1984) in between the two. In terms of d_w this gives

$$d_w = 2d_f - d_G. \tag{13}$$

for $d = 2$: $d_w = 73/24 \simeq 3.04$ (long-range limit) and $d_w \simeq 2.75 \pm 0.04$ (short-range limit).

Note that the excluded volume effect gets stronger in the short-range limit. For $p = 1$ case I get, surprisingly, $d_s = \frac{4}{3}$ (long-range limit) and $d_s = 2$ (short-range limit). For three-dimensional percolation ($d_f \simeq 2.5$) the preliminary results for d_G are $d_G \simeq 1.1$ (long-range) and $d_G \simeq 1.3$ (short-range); correspondingly, the crossover for dynamical exponents are $d_w \simeq 3.9$, $d_s \simeq 1.28$ (long-range) and $d_w \simeq 3.7$, $d_s \simeq 1.35$ (short-range). The crossover is not as sharp for $d = 3$. However, the short-range behaviour still appears to be different from the random diffusion on percolation. For $d_{\min}$ in the long-range limit I get, combining (13) and (8), $d_{\min} = d_w - d_f$. Note also that for the long-range limit one finds, combining (6), (3) and (12), $d_w/d_f = (1 + 1/d_l)$ for diffusion on percolation (compare with Havlin *et al* (1984) for transport on branched polymers). One can, of course, consider different ways of time rescaling for the 'butterfly' model.

In summary, I suggest the 'exact' value $d_G = 1/\nu$ for the long-range 'butterfly' model. The growth behaviour is found to be the same for the short-range 'butterfly' walk and a random walk on a Euclidean lattice, but different on a percolation cluster. The crossover is observed in the spectral dimension when the dynamic aspect is introduced to the model. Both long- and short-range limits of this growth-limited diffusion process are different from a random diffusion on percolation. I suggest a scaling relation for the 'chemical' dimension $d_l = d_f/(d_f - d_G)$ where $d_{\min} = d_f - d_G$. More numerical work is suggested to check the proposed scaling relations and a study of cluster perimeters is proposed for various diffusion processes, e.g. the 'looking ant' (Seifert and Suessenbach 1984).

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